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One-Parameter Flows with the Pseudo Orbit Tracing Property

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§ 1 Introduction

Let X be a compact metric space with metric d and f a homeomorphism of X. A sequence of points $\{x_i\}_{i\in\mathbb{Z}}$ is a δ-pseudo orbit of (X,f) if $d(fx_i, x_{i+1}) \leq \delta$ $(i\in\mathbb{Z})$. (X,f) is said to have the pseudo orbit tracing property (abbrev. P.O.T.P.) if for $\varepsilon>0$ there is $\delta>0$ such that for every δ-pseudo orbit $\{x_i\}_{i\in\mathbb{Z}}$ there is a point $x\in X$ with $d(f^i x, x_i) \leq \varepsilon$ $(i\in\mathbb{Z})$. (X,f) is expansive if there is $\delta>0$ such that $d(f^i x, f^i y) \leq \delta$ $(i\in\mathbb{Z})$ implies x=y. The P.O.T.P. and the expansiveness play important roles in several place in dynamics. N.Aoki [2] proved that if (X,f) has the P.O.T.P., then the restriction to its nonwandering set, (Ω,f), also has the P.O.T.P., and that if (Ω,f) has the P.O.T.P. and expansiveness then Ω splits into the finite disjoint union of closed invariant subsets on each of which f is topologically transitive. A.Morimoto [10] proved that if f is an isometry on a compact connected manifold M with dim $(M) \geq 1$, then f cannot have the P.O.T.P..

In this paper we introduce a concept of P.O.T.P. for 1-parameter flows, and investigate the properties of 1-parameter flow with the P.O.T.P.. For closed subsets Y_1 and Y_2 of X, put $d(Y_1,Y_2) = \inf\{d(y_1,y_2) : y_i\in Y_i, i=1,2\}$. Let $\phi: X\times\mathbb{R} \longrightarrow X$ be a flow on X; i.e. ϕ is continuous, $\phi(\cdot,0)$ is an identity map of X and $\phi(x,t+s) = \phi(\phi(x,t),s)$. Such a flow is often written by (X,ϕ). If Y is a subset of X and J is a subset of $\mathbb{R}$, we write $Y\cdot J = \phi(Y\times J)$. If Y is closed and invariant (i.e. $Y\cdot t = Y$ $(t\in\mathbb{R})$), (Y,ϕ) denotes the restriction of ϕ to $Y\times\mathbb{R}$. By $\Omega=\Omega(\phi)$ we denote the nonwandering set of ϕ; $\{x\in X$: for every open neighborhood U of x and every $T>0$, $U\cap(U\cdot[T,\infty)) \neq$ empty$\}$, which is closed and invariant.

Given $\delta>0$ and $T>0$, a (δ,T)-chain of (X,ϕ) is a collection $\{x_a,x_{a+1},\ldots,x_b;t_a,t_{a+1},\ldots,t_b\}$ $(x_i\in X,\ t_i\geq 0,\ a\leq i\leq b)$ ($a=-\infty$ and $b=\infty$ are permitted) such that $t_i \geq T$ and $d(x_i\cdot t_i, x_{i+1})\leq\delta$ $(a\leq i\leq b-1)$. A finite (δ,T)-chain $\{x_i;t_i\}_a^b$ $(-\infty<a\leq b<\infty)$ is naturally extended to an infinite (δ,T)-chain $\{x_i;t_i\}_{i\in\mathbb{Z}}$. Let x_0*t denote a point on a (δ,T)-chain t units time from x_0. More precisely, if $t\geq 0$ then $x_0*t = x_i\cdot(t-\sum_0^{i-1} t_n)$ where $\sum_0^{i-1} t_n \leq t < \sum_0^i t_n$; and if $t<0$ then $x_0*t = x_i\cdot(t+\sum_i^{-1} t_n)$ where $-\sum_i^{-1} t_n \leq t < -\sum_{i+1}^{-1} t_n$. Define $x_0*\mathbb{R}$ to be $\bigcup_{t\in\mathbb{R}} x_0*t$ and say it a (δ,T)-chain.

By analogy with the case of a homeomorphism, the first attempt to define pseudo orbit tracing properties for ϕ may be following; for every $\varepsilon>0$ there are $\delta>0$ and $T>0$ such that for every (δ,T)-chain $x_0*\mathbb{R}$ there is $x\in X$ with $d(x_0*t,\ x\cdot t)\leq\varepsilon$ $(t\in\mathbb{R})$. However there is an Axiom A flow (see [13]) the nonwandering set of which does not have this property (For example, the flow having a hyperbolic closed orbit as one of connected components of the nonwandering set). For this reason, we need to allow somewhat time lags to occur. To do this we define a notion of reparametrization as follow. Let Rep denote the set of increasing homeomorphisms of $\mathbb{R}$ fixing the origin. The element of Rep is called a reparametrization. Define a subset of Rep by

$$\mathrm{Rep}(\varepsilon) = \left\{g\in\mathrm{Rep}: \left|\frac{g(s)-g(t)}{s-t} - 1\right|\leq\varepsilon \quad (t\neq s)\right\} \quad (\varepsilon>0).$$

Given (δ,T)-chain x_0*R, if there are $y\in X$ and $g\in Rep(\varepsilon)$ such that $d(x_0*t, y\cdot g(t))\leq\varepsilon$ $(t\in\mathbb{R})$, then $x_0*\mathbb{R}$ is said to be ε-traced by (y,g) (or simply, by y). We remark that our definition of ε-tracing is different from one given by J.E.Franke and J.F.Selgrade [5,7]. We propose the following definition as the P.O.T.P. for flows.

Definition 1.1. A flow (X,ϕ) has the pseudo orbit tracing property (abbrov. P.O.T.P.) if for every $\varepsilon>0$ there are $\delta>0$ and $T>0$ such that every (δ,T)-chain is ε-traced by a pair (y,g) of some $y\in X$ and some $g\in Rep(\varepsilon)$.

This definition is clearly independent of the choice of metrics. Using Bowen's approximation theorem [3, Theorem 2.2], we have that the restriction of an Axiom A flow to (Ω,ϕ) has the P.O.T.P.. To see this, for $\{t_i\}_{i\in\mathbb{Z}}$ $(t_i>0)$ define $u_i\in\mathbb{R}$ by $u_0=0$, $u_i=\sum_0^{i-1} t_j$ $(i>0)$ and $u_i=-\sum_i^{-1} t_j$ $(i<0)$, and set $STEP(\varepsilon,\{t_i\})=\{s:\mathbb{R}\rightarrow\mathbb{R}\mid$ s is constant on (u_i,u_{i+1}), $s(u_i)=s(u_i+0)$ or $s(u_i-0)$, $|s(u_0)|\leq\varepsilon$ and $|s(u_i+0)-s(u_i-0)|\leq\varepsilon\}$. Bowen's theorem implies that for $\varepsilon>0$ there are $\delta>0$ and $T\geq 1$ such that for every (δ,T)-chain $\{x_i;t_i\}_{i\in\mathbb{Z}}$ there are $x\in X$ and $s\in STEP(\varepsilon,\{t_i\})$ with $d(x_0*t, x\cdot(t+s(t)))\leq\varepsilon$ $(t\in R)$. We define $g(u_0)=0$, $g(u_i)=u_i+s(u_i-0)$ $(i\neq 0)$ and extend g linearly between these points, then $g\in Rep(3\varepsilon)$. By uniform continuity of ϕ, we get the required conclusion.

In §2 we give the equivalent definitions of P.O.T.P.(Theorem 1). And in §§3 and 4, we prove that if (X,ϕ) has the P.O.T.P. then so has (Ω,ϕ) (Theorem 2), and also that if (Ω,ϕ) has the P.O.T.P. and expansiveness then the spectral decomposition of Ω is obtained (Theorem 3). Further examples of flows with the P.O.T.P. are provided by suspensions of homeomorphisms with the P.O.T.P.. In fact, using the invariance of the P.O.T.P. under some equivalence for flows (Theorem 4), we prove that a suspension of a homeomorphism has the P.O.T.P. if and only if the homeomorphism has the P.O.T.P. (Theorem 5). When ϕ is an isometric flow on a compact connected manifold, if ϕ has the P.O.T.P. then ϕ must be minimal (Theorem 6). From Theorems 5 and 6, we show that in general the direct product of flows with the P.O.T.P. need not to have the P.O.T.P.(Remark 4), and that the time T-map of flow with the P.O.T.P. need not to have the P.O.T.P.(Remark 5).

§ 2 Equivalent definitions.

In this section we state the properties of Rep(ε) and the equivarent definitions of the P.O.T.P..

Lemma 2.1. (1) For $g \in \mathrm{Rep}(\varepsilon)$ and a constant $c \in \mathbb{R}$, define $h(t) = g(c+t)-g(c)$ $(t \in \mathbb{R})$, then $h \in \mathrm{Rep}(\varepsilon)$.

(2) For a sequence $\{g_n\}_{n \geq 0}$ of Rep(ε) there is $g \in \mathrm{Rep}(\varepsilon)$ such that for every $N>0$ there is a sequence $\{n(i)\} \subset \{n\}$ so that

$g_{n(i)}|_{[-N,N]} \longrightarrow g|_{[-N,N]}$ (uniformly as $i \to \infty$).

Proof. It is clear that h is an increasing homeomorphism of $\mathbb{R}$ with $h(0)=0$. Since $\left|\frac{h(t)-h(s)}{t-s} - 1\right| = \left|\frac{g(c+t)-g(c+s)}{(c+t)-(c+s)} - 1\right| \leq \varepsilon$ $(t \neq s)$, we have $h \in \mathrm{Rep}(\varepsilon)$. (1) is proved.

For every $N>0$, $\{g_n|_{[-N,N]}\}_{n \geq 0}$ is uniformly bounded and equicontinuous. Hence, using the Ascoli-Arzela's theorem and σ-compactness of $\mathbb{R}$, we get an increasing homeomorphism $g: \mathbb{R} \longrightarrow \mathbb{R}$ such that for every $N>0$ there is a subsequence $\{n(i)\}$ with $g_{n(i)}|_{[-N,N]} \longrightarrow g|_{[-N,N]}$ (uniformly as $i \longrightarrow \infty$). Let $t,s \in \mathbb{R}$ $(t \neq s)$ be given and take $N>0$ with $t,s \in [-N,N]$. Since $\left|\frac{g(t)-g(s)}{t-s} - 1\right| =$

$$\lim_{i\to\infty} \left|\frac{g_{n(i)}(t)-g_{n(i)}(s)}{t-s} - 1\right| \leq \varepsilon \quad \text{and } g(0) = \lim_{i\to\infty} g_{n(i)}(0) = 0,$$

we have $g \in \mathrm{Rep}(\varepsilon)$. (2) is proved.

<u>Theorem 1.</u> The following are equivlent for a flow ϕ on X.

(a) (X,ϕ) has the P.O.T.P..

(b) for every $\varepsilon>0$ and every $T>0$ there is $\delta>0$ such that (δ,T)-chain is ε-traced by a pair (y,g) of some $y \in X$ and $g \in \mathrm{Rep}(\varepsilon)$.

(c) for every $\varepsilon>0$ there are $\delta>0$ and $T>0$ such that for every integer $k>0$ and every finite (δ,T)-chain $\{x_i;t_i\}_0^k$ there are $x \in X$ and $g \in \mathrm{Rep}(\varepsilon)$ with $d(x_0*t, x\cdot g(t)) \leqq \varepsilon$ $(t \in [0, \sum_0^k t_i])$.

<u>Proof.</u> Clearly (b) implies (a). We show that (a) implies (b). Let $\varepsilon>0$ and $T>0$ be given. By assumption there are $\delta_0>0$ and $T_0>0$ such that every (δ_0,T_0)-chain is $\varepsilon/2$-traced. If $T \geqq T_0$, the statement is clear because a (δ_0,T)-chain is a (δ_0,T_0)-chain. There is an integer $M>0$ with $TM \geqq T_0$ when $T < T_0$. By continuity of ϕ there is $\delta>0$ such that if $x_0*[0,T_0]$ is (δ,T)-chain having at most M+1 jumps then $d(x_0*t, x_0\cdot t) < \delta_0$ $(t \in [0,T_0])$. Let $x_0*\mathbb{R}$ be a (δ,T)-chain. Since $\{x_0*(iT_0);t_i=T_0\}_{i\in\mathbb{Z}}$ is a (δ_0,T_0)-chain, there are $y \in X$ and $g \in \mathrm{Rep}(\varepsilon/2)$ with $d((x_0*(iT_0))\cdot t,\ y\cdot g(iT_0+t)) < \varepsilon/2$ $(t \in [0,T_0),\ i \in \mathbb{Z})$. Hence we have $d(x_0*(iT_0+t), y\cdot g(iT_0+t)) \leqq d(x_0*(iT_0+t), (x_0*(iT_0))\cdot t)$ $+ d((x_0*(iT_0))\cdot t,\ y\cdot g(iT_0+t)) < \varepsilon$ for $t \in [0,T_0]$ and $i \in \mathbb{Z}$; i.e. $x_0*\mathbb{R}$ is ε-traced by (y,g).

We have shown that (a) and (b) are equivalent, thus it remains to show (a) and (c) are equivalent. It is clear that (a) implies (c). To show (c) implies (a), let $\varepsilon>0$ be given and take $\delta = \delta(\varepsilon) > 0$ and $T = T(\varepsilon) > 0$ as in (c). Let $\{x_i;t_i\}_{i\in\mathbb{Z}}$ be a (δ,T)-chain. Then for every $n>0$ there are $y_n \in X$ and $g_n \in \mathrm{Rep}(\varepsilon)$ such that $d((x_0*(-n))*t,\ y_n\cdot g_n(t)) \leqq \varepsilon$ $(t \in [0,2n])$. Since X is compact, $y_n\cdot g_n(n) \longrightarrow y$ $(n\to\infty)$ taking subsequence if necessary. For $h_n(t) = g_n(n+t) - g_n(n)$ we have

$h_n \in \mathrm{Rep}(\varepsilon)$ by Lemma 2.1(1). By Lemma 2.1(2) there is $h \in \mathrm{Rep}(\varepsilon)$ such that for every $N>0$ there is a sequence $\{m\}$ with $h_{m|[-N,N]} \longrightarrow h_{|[-N,N]}$ (uniformly as $m \to \infty$). Let $t \in \mathbb{R}$ be given. Since there is a sequence $\{m\}$ with $h_m(t) \longrightarrow h(t)$ $(m \to \infty)$, it follows that

$$d(x_0*t, y\cdot h(t)) = \lim_{m\to\infty} d(x_0*t, (y_m\cdot g_m(m))\cdot h_m(t))$$

$$= \lim_{m\to\infty} d((x_0*(-m))*(m+t), y_m\cdot g_m(m+t)) \le \varepsilon ,$$

i.e. x_0*R is ε-traced by (y,h). The proof is completed.

§ 3 The chain recurrent sets of flows with the P.O.T.P..

Let ϕ be a flow on X. Given $\delta>0$ and $T>0$. For $x,y \in X$, x is (δ,T)-related to y (written $x \xleftrightarrow{\delta,T} y$) if there are (δ,T)-chains $\{x_i;t_i\}_0^m$ and $\{y_i;s_i\}_0^n$ with $x=x_0=y_n$ and $y=y_0=x_m$. If $x \xleftrightarrow{\delta,T} y$ for every $\delta>0$ and every $T>0$, then x is related to y (written $x \sim y$). The chain recurrent set of ϕ , R, is $\{x\in X\colon x \sim x\}$. Clearly "$\xleftrightarrow{\delta,T}$" and " $\sim$ " are equivarence relations of R. In Lemma 3.1(4) it will be proved that R is a closed invariant set containing Ω. In this section we prove following

Theorem 2. Let ϕ be a flow on X. If (X,ϕ) has the P.O.T.P., then $R = \Omega$, and (Ω,ϕ) has the P.O.T.P..

Lemma 3.1. (1) If $x,y\in R$ with $d(x,y)<\delta$, then $x \xleftrightarrow{\delta,T} y$ for every $T>0$.

(2) For every $\delta>0$ and every $T>0$, there is a $\gamma>0$ such that if $y\in X$ holds $d(x,y)<\gamma$ for some $x\in R$ then $y \xleftrightarrow{\delta,T} y$.

(3) For every $x\in R$ and every $\tau\in R$, x is related to $x\cdot\tau$.

(4) R is a closed invariant set containing Ω.

Proof. The proof of (1). Put $\alpha=\delta-d(x,y)$. Since $x,y\in R$, for every $T>0$ there are (α,T)-chains $\{x_i;t_i\}_0^m$ and $\{y_i;s_i\}_0^n$ with $x_0=x=x_m$ and $y_0=y=y_n$. Then $\{x_0,\dots,x_{m-1},y;t_0,\dots,t_m\}$ (resp. $\{y_0,\dots,y_{n-1},x;s_0,\dots,s_n\}$) is a (δ,T)-chain from x to y (resp. from y to x), and so $x\xleftrightarrow{\delta,T}y$.

The proof of (2). For every $\delta>0$ and every $T>0$, there is $\delta/2>\gamma>0$ such that $d(x,y)<\gamma$ implies $d(x\cdot T,y\cdot T)<\delta$. Since $x\in R$, there is a $(\delta/2,2T)$-chain $\{x_i;t_i\}_0^k$ with $x_0=x=x_k$. Then $\{y,x\cdot T,x_1,\dots,x_k;T,t_0-T,t_1,\dots,t_k\}$ is a (δ,T)-chain from y to itself. Thus (2) is obtained.

The proof of (3). Let $x\in R$ and $\tau\in\mathbb{R}$ be given. For every $\delta>0$ there is $\delta>\gamma>0$ such that $d(x,y)<\gamma$ implies $d(x\cdot\tau,y\cdot\tau)<\delta$. Put $S=T+|\tau|$. Since $x\in R$, there is a (γ,S)-chain $\{x_i;t_i\}_0^k$ with $x_0=x=x_k$. Then $\{x\cdot\tau,x_1,\dots,x_k;t_0-\tau,t_1,\dots,t_k\}$ is a (δ,T)-chain from $x\cdot\tau$ to x. Also $\{x_0,\dots,x_{k-1},x\cdot\tau;t_0,\dots,t_{k-2},t_{k-1}+\tau,t_k\}$ is a (δ,T)-chain from x to $x\cdot\tau$, and so $x\xleftrightarrow{\delta,T}x$. Since δ and T are arbitrary, we get $x\sim x$.

The proof of (4). For $x\in R$ and $\tau\in\mathbb{R}$, by (3) we have $x\cdot\tau\sim x\cdot\tau$ and hence R is invariant. If $y\in\bar{R}$ then by (2) we have $y\xleftrightarrow{\delta,T}y$ for every $\delta>0$ and every $T>0$. Hence $y\in R$ and so R is closed. To see $\Omega\subset R$, let $x\in\Omega$ be given. For every $\delta>0$ and every $T>0$, there is $\delta>\gamma>0$ such that $d(x,y)<\gamma$ implies $d(x\cdot T,y\cdot T)<\delta$. Since $x\in\Omega$, there is $y\in X$ with $d(x,y)<\gamma$ and $d(x,y\cdot S)<\gamma$ for some $S\geq 2T$. Then $\{x,y\cdot T,x;T,S-T,0\}$ is a (δ,T)-chain from x to itself. Since δ and T are arbitrary, we have $x\in R$.

Lemma 3.2. Given $\delta>0$ and $T>0$, R is splited into equivalence classes A_λ under the (δ,T)-relation; i.e. $R=\bigcup_\lambda A_\lambda$. Then each A_λ

is an invariant, open and closed subset in R.

Proof. For $x \in A_\lambda$ and $\tau \in \mathbb{R}$, by Lemma 3.1(3) we have $x \xleftrightarrow{\delta,T} x\cdot\tau$ and so $x\cdot\tau \in A_\lambda$. Hence A_λ is invariant. By Lemma 3.1(1), $\{y \in R: d(x,y) < \delta,\ x \in A_\lambda\} \subset A_\lambda$, that is, A_λ is open in R. Since $A_\lambda = R - \bigcup_{\lambda\neq\mu} A_\mu$, A_λ is closed in R.

From Lemma 3.2, $\{A_\lambda\}$ is finite because R is compact. Hence $R = \bigcup_1^m A_i$ for some $m = m(\delta,T) > 0$.

Proof of Theorem. Let us assume that (X,ϕ) has the P.O.T.P.. First we show that $R = \Omega$. Since $\Omega \subset R$ by Lemma 3.1(4), we must show $R \subset \Omega$. To do this, let $x \in R$ and $\varepsilon > 0$ be given. Put $U = \{y \in X: d(x,y) \leqq \varepsilon\}$. By assumption, there are $\delta > 0$ and $T > 0$ such that for every (δ,T)-chain $\{x_i;t_i\}_{i\in \mathbb{Z}}$ in X there are $y \in X$ and $g \in \mathrm{Rep}(\varepsilon)$ with $d(x_0 * t, y\cdot g(t)) \leqq \varepsilon$ $(t \in \mathbb{R})$. Since $x \in R$, there is a (δ,T)-chain $\{x_i;t_i\}_0^k$ with $x_0 = x = x_k$. Put $x_{kn+i} = x_i$ and $t_{kn+i} = t_i$ for $n \in \mathbb{Z}$, then $\{x_i;t_i\}_{i\in\mathbb{Z}}$ is a (δ,T)-chain. Then there are $y \in X$ and $g \in \mathrm{Rep}(\varepsilon)$ with $d(x_0 * t, y\cdot g(t)) \leqq \varepsilon$ $(t \in \mathbb{R})$. Since $g(0) = 0$, we get $y \in U$. For $n_j = j\sum_0^{k-1} t_i$, it follows that $g(n_j) \geqq (1-\varepsilon)n_j \uparrow \infty$ and $d(x, y\cdot g(n_j)) = d(x_0 * n_j, y\cdot g(n_j)) \leqq \varepsilon$. Therfore $x \in \Omega$.

To see that (Ω,ϕ) has the P.O.T.P., let $\varepsilon > 0$ be given. In order to our conclusion, by Theorem 1(c) it is enough to show that there are $\delta > 0$ and $T > 0$ such that for every finite (δ,T)-chain $\{\bar{x}_i;t_i\}_0^k$ in R there are $y \in R$ and $g \in \mathrm{Rep}(\varepsilon)$ with $d(x_0 * t, y\cdot g(t)) \leqq \varepsilon$ $(t \in [0, \sum_0^k t_i])$. By assumption, there are $\gamma > 0$ and $T > 0$ such that (γ,T)-chain in X is ε-traced by some point in X. Let $R = \bigcup_1^m A_i$ be the splitting of R into equivarence classes under (γ,T)-relation. Since each A_i is closed in X (by Lemmas 3.1(4) and 3.2), we have

$\delta_1 = \min\{d(A_i,A_j): i\neq j\} > 0$. Take $0<\delta<\min\{\gamma,\delta_1\}$ and let $\{x_i;t_i\}_0^k$ be a finite (δ,T)-chain in R. Without loss of generality, we may assume $x_0 \in A_1$. Since $A_1\cdot\mathbb{R} = A_1$ (by Lemma 3.2) and $d(x_i\cdot t_i,x_{i+1})<\delta$ $<\delta_1$, we have $\{x_i\}_0^k \subset A_1$. Since $x_0 \xleftrightarrow{\gamma,T} x_k$, there is a (γ,T)-chain $\{x_0',\ldots,x_a';t_0',\ldots,t_a'\}$ in X with $x_0' = x_k$ and $x_a' = x_0$. For $n\in\mathbb{Z}$, put

$$z_i = \begin{cases} x_{i-n(k+a)} & n(k+a) \leq i < n(k+a)+k \\ x'_{i-n(k+a)-k} & n(k+a)+k \leq i < (n+1)(k+a), \end{cases}$$

$$s_i = \begin{cases} t_{i-n(k+a)} & n(k+a) \leq i < n(k+a)+k \\ t'_{i-n(k+a)-k} & n(k+a)+k \leq i < (n+1)(k+a). \end{cases}$$

Then $\{z_i;s_i\}_{i\in\mathbb{Z}}$ is a (γ,T)-chain in X, so there are $z\in X$ and $h\in\mathrm{Rep}(\varepsilon)$ with $d(z_0*t,z\cdot h(t))\leq\varepsilon$ $(t\in\mathbb{R})$. Put $p = \sum_0^{k+a-1} s_i$ and suppose a sequence $\{z\cdot h(np)\}_{n\geq 0}$. When $\{z\cdot h(np)\}_{n\geq 0}$ is finite, z is periodic point, so $z\in\Omega= R$. Hence $(y,g) = (z,h)$ is the required tracing pair. If $\{z\cdot h(np)\}_{n\geq 0}$ is infinite, a subsequence converges to some point $y\in X$. It is clear that $y\in R$. Define $g_n\in \mathrm{Rep}(\varepsilon)$ by $g_n(t) = h(np+t)-h(np)$ $(t\in\mathbb{R})$. By Lemma 2.1(2) there are $g\in\mathrm{Rep}(\varepsilon)$ and a sequence $\{n'\}$ such that $z\cdot h(n'p)\longrightarrow y$ $(n'\longrightarrow\infty)$, and such that $g_{n'}|[0,p]$ uniformly converges to $g|[0,p]$ $(n'\longrightarrow\infty)$. Then for $t\in[0, \sum_0^{k-1} t_i]\subset[0,p]$, we get

$$\begin{aligned} d(x_0*t,y\cdot g(t)) &= \lim_{n'\to\infty} d(x_0*(n'p+t),(z\cdot h(n'p))\cdot g_{n'}(t)) \\ &= \lim_{n'\to\infty} d(x_0*(n'p+t),z\cdot h(n'p+t)) \leq \varepsilon. \end{aligned}$$

The proof is completed.

§ 4 Spectral decomposition of Ω.

Let (X,ϕ) be a flow on compact metric space. (X,ϕ) is said to be expansive([4]) if for any $\varepsilon>0$ there is $\eta=\eta(\varepsilon)>0$ such that for some $x,y\in X$ and for a continuous map $g:\mathbb{R}\longrightarrow\mathbb{R}$ with $g(0)=0$, if $d(x\cdot t,y\cdot g(t))\leqq\eta$ $(t\in\mathbb{R})$ then $x\in y\cdot[-\varepsilon,\varepsilon]$. $\eta=\eta(\varepsilon)$ is called an expansive constant corresponding with ε.

Lemma 4.1 (Lemma 1 in [4]) If (X,ϕ) is expansive, then each fixed point of ϕ is an isolated point of X.

Lemma 4.2 Assume that (X,ϕ) is expansive without fixed point. For $\varepsilon>0$, let $\eta=\eta(\varepsilon)>0$ be an expansive constant corresponding with ε. If for $x,y\in X$ and $g\in \mathrm{Rep}(\eta)$, $d(x\cdot t,y\cdot g(t))\leqq\eta$ for $t\geqq 0$ (resp. $t\leqq 0$), then for every $\gamma>0$ there is $N>0$ such that $d(x\cdot t,(y\cdot[-\varepsilon,\varepsilon])\cdot g(t))\leqq\gamma$ for $t\geqq N$ (resp. $t\leqq -N$).

Proof. If the assertion is false, there is $\gamma>0$ so that for every $n>0$ there are $x_n,y_n\in X$, $g_n\in \mathrm{Rep}(\eta)$ and $t_n\geqq n$ such that

$$d(x_n\cdot t,y_n\cdot g_n(t))\leqq\eta \ (t\geqq 0) \text{ and } d(x_n\cdot t_n,(y_n\cdot[-\varepsilon,\varepsilon])\cdot g_n(t_n)) > \gamma .$$

Without loss of generality, we may assume that $x_n\cdot t_n\longrightarrow x$ and $y_n\cdot g_n(t_n)\longrightarrow y$. Put $h_n(t) = g_n(t_n+t)-g_n(t_n)$, then $h_n\in \mathrm{Rep}(\eta)$ by Lemma 2.1(1). By Lemma 2.1(2) there is $h\in \mathrm{Rep}(\eta)$ such that for every $N>0$ there is a sequence $\{m\}$ so that $h_m|[-N,N]\longrightarrow h|[-N,N]$ (uniformly). Let $t\in\mathbb{R}$ be given. There are $N>0$ with $t\in[-N,N]$ and a sequence $\{m\}$ such that $h_m(t)\longrightarrow h(t)$ $(m\longrightarrow\infty)$ and $-t_m\leqq t$. Then we get $d(x\cdot t,y\cdot g(t)) = \lim_{m\to\infty} d((x_m\cdot t_m)\cdot t,(y_m\cdot g_m(t_m))\cdot h_m(t))$ $= \lim_{m\to\infty} d(x_m\cdot s_m,y_m\cdot g_m(s_m)) \leqq \eta$ where $s_m = t+t_m \geqq 0$. Since $t\in\mathbb{R}$ is arbitrary, by expansiveness $x\in y\cdot[-\varepsilon,\varepsilon]$. But for every $s\in[-\varepsilon,\varepsilon]$, $d(x,y\cdot s) = \lim_{n\to\infty} d(x_n\cdot t_n,(y_n\cdot g_n(t_n))\cdot s)\geqq\gamma$;i.e. $x\notin y\cdot[-\varepsilon,\varepsilon]$. This is a contradiction as well.

Teorem 3 (Spectral decomposition theorem). Let Ω be the non-wandering set of a flow (X,ϕ). Assume that (Ω,ϕ) is expansive and has the P.O.T.P.. Then Ω is uniquely expressed as a finite disjoint union $\Omega = \bigcup_{i=1}^{a}\{\omega_i\} \cup \bigcup_{i=1}^{b} \Omega_i$ where $\{\omega_i\}_{i=1}^{a}$ is the set of fixed points of ϕ and Ω_i $(1 \leq i \leq b)$ is a closed invariant set such that (Ω_i,ϕ) is topologically transitive.

The proof of Theorem is done along the following steps (S.1)~(S.4). After this we assume (Ω,ϕ) is expansive and has the P.O.T.P..

(S.1) Denote by $\mathrm{Per}(\phi)$ the set of periodic points; i.e. $\{x \in X: x \cdot t = x \text{ for some } t \neq 0\}$. Then $\mathrm{Per}(\phi)$ is dence in Ω.

Proof. This is easily obtained by the P.O.T.P. and expansivness.

(S.2) Let $\{B_\lambda\}$ be the equivalence classes of Ω under the relation " ~ ". Then each B_λ is invariant, open and closed in Ω.

Proof. Since Ω is invariant, so is B_λ by Lemma 3.1(3). By Lemma 3.1(2), B_λ is closed in Ω. We show that B_λ is open in Ω. Let F be the set of fixed points of ϕ. By Lemma 4.1 each element of F is isolated in Ω, so that F is finite. If B_λ is contained in F, clearly B_λ is a single point and so B_λ is open in Ω. Suppose that B_λ is not contained in F. Since F is open and closed in Ω, $(\Omega - F,\phi)$ has the P.O.T.P. and expansiveness. So we can suppose that (Ω,ϕ) has no fixed point and $B_\lambda \subset \Omega$.

Let $\varepsilon>0$ be given. By expansiveness there is an expansive constant $\eta=\eta(\varepsilon)>0$. By the P.O.T.P. there are $\delta>0$ and $T>0$ such that every (δ,T)-chain in Ω is η-traced by some point in Ω. Put $U_\delta(B_\lambda) = \{y \in \Omega: d(y,B_\lambda)<\delta\}$ and take $p \in U_\delta(B_\lambda) \cap \mathrm{Per}(\phi)$. There is $\tau>T$ such that $p \cdot \tau = p$. Take $y \in B_\lambda$ with $d(y,p)<\delta$.

Since a collection $\{\ldots,p,p,y,y\cdot T,y\cdot(2T),\ldots;\ldots,\tau,\tau,T,T,T,\ldots\}$ is a (δ,T)-chain, there are $x \in \Omega$ and $g \in \mathrm{Rep}(\eta)$ such that

$d(p\cdot t, x\cdot g(t)) \leq \eta$ $(t \leq 0)$ and $d(y\cdot t, x\cdot g(t)) \leq \eta$ $(t \geq 0)$. For every $\gamma > 0$ and $S > 0$, by Lemma 4.2 there is $N > S$ so that $d(p\cdot t, (x\cdot[-\varepsilon,\varepsilon])\cdot g(t)) \leq \gamma$ $(t \leq -N)$ and $d(y\cdot t, (x\cdot[-\varepsilon,\varepsilon])\cdot g(t)) \leq \eta$ $(t \geq N)$. Take $K \geq N$ with $p\cdot K = P$. There are $s, s' \in [-\varepsilon,\varepsilon]$ such that $d(p, x\cdot(s+g(-K))) = d(p\cdot(-K), (x\cdot s)\cdot g(-K)) \leq \gamma$ and $d(y\cdot N, x\cdot(s'+g(N))) \leq \gamma$. By Lemma 3.1(1), $p \xleftrightarrow{\gamma, S} x\cdot(s+g(-K))$ and $x\cdot(s'+g(N)) \xleftrightarrow{\gamma, S} y\cdot N$. Since $x\cdot(s+g(-K)) \sim x\cdot(s'+g(N))$ and $y\cdot N \sim y$ (by Lemma 3.1(3)), we have $p \xleftrightarrow{\gamma, S} y$. Since γ and S are arbitrary, we obtain $p \in B_\lambda$. Therefore by (S.1) we get $B_\lambda \supset \overline{U_\delta(B_\lambda) \cap \mathrm{Per}(\phi)} \supset U_\delta(B_\lambda) \cap \overline{\mathrm{Per}(\phi)} = U_\delta(B_\lambda)$.

Since Ω is compact, by (S.2), Ω is uniqly expressed as a disjoint union $\Omega = \bigcup_{i=1}^{m} B_i = \bigcup_{i=1}^{a} \omega_i \cup \bigcup_{i=1}^{b} \Omega_i$ $(a+b=m)$ where ω_i $(1 \leq i \leq a)$ is a fixed point and Ω_i $(1 \leq i \leq b)$ is an equivalence class under "$\sim$" without fixed point.

(S.3) (B_i, ϕ) has the P.O.T.P. $(1 \leq i \leq m)$.

<u>Proof.</u> Put $\alpha = \min\{d(B_i, B_j) : i \neq j\}$ and let $\alpha > \varepsilon > 0$ be given. Since (Ω, ϕ) has the P.O.T.P., there are $\delta > 0$ and $T > 0$ such that every (δ, T)-chain in Ω is ε-traced by some point in Ω. If a (δ, T)-chain in B_i is ε-traced by $y \in \Omega$, then we have $y \in B_i$ by the choice of ε. Hence (B_i, ϕ) has the P.O.T.P..

It is clear that (ω_i, ϕ) $(1 \leq i \leq a)$ is topologically transitive.

(S.4) (Ω_i, ϕ) is topologically transitive $(1 \leq i \leq b)$.

<u>Proof.</u> Let U and V be open set in Ω_i. There is $\varepsilon > 0$ such that $U_\varepsilon(x) \subset U$ and $U_\varepsilon(y) \subset V$ for some $x \in U$ and $y \in V$ (where $U_\varepsilon(z) = \{w \in \Omega_i : d(z,w) \leq \varepsilon\}$). By (S.3) there are $\delta > 0$ and $T > 0$ so that every (δ, T)-chain in Ω_i is ε-traced by some point in Ω_i. Since $x \sim y$, there is a (δ, T)-chain $\{y_i; t_i\}_0^k$ with $y_0 = y$ and $y_k = x$. Then there are $z \in U_\varepsilon(y)$ and $K > 0$ with $d(x, z\cdot K) \leq \varepsilon$. So $z \in U_\varepsilon(x)\cdot(-K) \cap U_\varepsilon(y) \subset U\cdot(-K) \cap V$ $\neq$ empty. The proof is completed.

§ 5 Topologically Lipschitz equivalence and suspensions.

Let X_i $(i=1,2)$ be a compact metric space with a metric d_i $(i=1,2)$ and let ϕ_i be a flow on X_i $(i=1,2)$. Suppose that (X_1,ϕ_1) is topologically equivalent to (X_2,ϕ_2); that is, there is a homeomorphism $\mu:X_1 \longrightarrow X_2$ such that μ maps orbits of (X_1,ϕ_1) onto orbits of (X_2,ϕ_2). In general μ dose not preserve the notion of (δ,T)-chain on (X_1,ϕ_1). For example, put $X_1 = X_2 = \{z \in \mathbb{C}: |z| \leq 1\}$, $\phi_1(z,t) = ze^{2\pi it}$ and $\phi_2(z,t) = ze^{2\pi it/|z|}$ $(z \neq 0)$, $=0$ $(z=0)$ where $i=\sqrt{-1}$. Since $\mu = id.:X_1 \longrightarrow X_2$ is a homeomorphism mapping orbits of (X_1,ϕ_1) onto orbits of (X_2,ϕ_2), (X_1,ϕ_1) is topologically equivalent to (X_2,ϕ_2). A collection $\{2^{-1},2^{-2},\ldots;1,1,\ldots\}$ is a $(2^{-1},1)$-chain of (X_1,ϕ_1), and the image of this chain under μ is $\{2^{-1},2^{-2},\ldots;2^{-1},2^{-2},\ldots\}$, which can not be a $(2^{-1},T)$-chain of (X_2,ϕ_2) for every $T>0$. Therefore we suppose the equivalent relation stronger than the topologically equivalence.

Definition 5.1. Let (X_i,ϕ_i) be a flow $(i=1,2)$. (X_1,ϕ_1) and (X_2,ϕ_2) are topologically Lipshitz equivalent if there are a homeomorphism $\mu:X_1 \rightarrow X_2$ and a continuous map $\sigma:X_1 \times \mathbb{R} \rightarrow \mathbb{R}$ with $\sigma(x,0)=0$ $(x \in X_1)$ such that for some $M \geqq m > 0$,

$$m \leqq \frac{\sigma(x,t)-\sigma(x,s)}{t-s} \leqq M \quad (x \in X_1,\ t \neq s) \quad \text{and}$$

$$\mu\phi_1(x,t) = \phi_2(\mu x, \sigma(x,t)) \quad (x \in X_1,\ t \in \mathbb{R}).$$

Clearly this is a equivalence relation. If σ is a continuous map as above, $\sigma_x = \sigma(x,\cdot):\mathbb{R} \rightarrow \mathbb{R}$ $(x \in X_1)$ is an increasing homeomorphism of $\mathbb{R}$ such that σ_x and σ_x^{-1} are Lipschitz continuous.

Lemma 5.2. Let $g \in \mathrm{Rep}(\varepsilon)$. If $\sigma_i : \mathbb{R} \longrightarrow \mathbb{R}$ $(i=1,2)$ is an increasing homeomorphism of $\mathbb{R}$ such that $\sigma_i(0) = 0$ and

$$m \leq \frac{\sigma_i(t)-\sigma_i(s)}{t-s} \leq M \quad (t \neq s) \text{ for some } M \geq m > 0,$$

then $h = \sigma_1^{-1} \circ g \circ \sigma_2 : \mathbb{R} \longrightarrow \mathbb{R}$ belongs to $\mathrm{Rep}(\frac{M}{m}\varepsilon)$.

Proof. Since σ_i $(i=1,2)$ and g are all increasing homeomorphisms of $\mathbb{R}$ fixing the origin, so is h; i.e. $h \in \mathrm{Rep}$. As

$$\left|\frac{h(t)-h(s)}{t-s} - 1\right| = \left|\frac{h(t)-h(s)-t+s}{\sigma_1 h(t)-\sigma_1 h(s)-\sigma_2(t)+\sigma_2(s)}\right| \cdot \left|\frac{g\sigma_2(t)-g\sigma_2(s)}{\sigma_2(t)-\sigma_2(s)} + \frac{-\sigma_2(t)+\sigma_2(s)}{\sigma_2(t)-\sigma_2(s)}\right| \cdot \left|\frac{\sigma_2(t)-\sigma_2(s)}{t-s}\right| \leq \frac{h(t)-h(s)-t+s}{m(h(t)-h(s)-t+s)} \cdot \varepsilon \cdot M = \frac{M}{m}\varepsilon \quad (t \neq s),$$

we get $h \in \mathrm{Rep}(\frac{M}{m}\varepsilon)$.

Theorem 4. Let (X_i, ϕ_i) $(i=1,2)$ be a flow. Assume that (X_1, ϕ_1) and (X_2, ϕ_2) are topologically Lipschitz equivalent. If (X_2, ϕ_2) has the P.O.T.P., then so does (X_1, ϕ_1).

Proof. By assumption there are a homeomorphism $\mu : X_1 \longrightarrow X_2$ and a continuous map $\sigma : X_1 \times \mathbb{R} \to \mathbb{R}$ such that $\sigma(x,0)=0$, $m \leq \frac{\sigma(x,t)-\sigma(x,s)}{t-s} \leq M$ $(x \in X_1,\ t \neq s)$ for some $M \geq m > 0$ and $\phi_1(x,t) = \mu^{-1}\phi_2(\mu x, \sigma(x,t))$ $(x \in X_1, t \in \mathbb{R})$. Let $\varepsilon > 0$ be given. By the uniform continuity of μ there is $\varepsilon' > 0$ so that $d_2(y,y') < \varepsilon'$ $(y,y' \in X_2)$ implies $d_1(\mu^{-1}y, \mu^{-1}y') < \varepsilon$. Take $\varepsilon' > \gamma > 0$ with $\frac{M}{m}\gamma < \varepsilon$. Then there are $\delta' > 0$ and $T' > 0$ such that (δ', T')-chain of (X_2, ϕ_2) is γ-traced. We take $\delta > 0$ with $d_1(x,x') < \delta$ $(x, x' \in X_1)$ implies $d_2(\mu x, \mu x') < \delta'$, and put $T = T'/m$. In order to get conclusion, by Theorem 1(c) it is enough to show that every finite (δ, T)-chain $\{x_i ; t_i\}_0^k$ of (X_1, ϕ_1) is ε-traced.

Since $d_2(\phi_2(\mu x_i, \sigma(x_i, t_i)), \mu x_{i+1}) = d_2(\mu\phi_1(x_i, t_i), \mu x_{i+1}) < \delta'$

and $\sigma(x_i,t_i)\geq mT=T'$, a collection $\{\mu x_i;\sigma(x_i,t_i)\}_0^k$ is a (δ',T')-chain of (X_2,ϕ_2). Then there are $y\in X_2$ and $g\in \mathrm{Rep}(\gamma)$ with $d_2((\mu x_0)*t,y\cdot g(t))\leq\gamma$ $(t\in[0,\sum_0^k\sigma(x_i,t_i)]\ )$. Put $x=\mu^{-1}y$.
To find a reparametrization we define an increasing homeomorphism $\tilde{\sigma}:\mathbb{R}\longrightarrow\mathbb{R}$ by

$$\tilde{\sigma}(t)=\begin{cases} t & (t\leqq 0)\\ \sigma(x_i,\tau)+\sum_0^{i-1}\sigma(x_j,t_j) & (\tau=t-\sum_0^{i-1}t_j\in(0,t_i],\ 0\leq i\leq k\ )\\ t+\sum_0^k\{\sigma(x_j,t_j)-t_j\} & (\ \sum_0^k t_j<t\ ),\end{cases}$$

where $\sum_0^{-1}t_j=0$. $\tilde{\sigma}$ satisfies that $\tilde{\sigma}(0)=0$ and $m\leq\frac{\tilde{\sigma}(t)-\tilde{\sigma}(s)}{t-s}\leq M$ $(t\neq s)$, and so by Lemma 5.2, $h=\sigma(x,\cdot)^{-1}\circ g\circ\tilde{\sigma}$ belongs in $\mathrm{Rep}(\frac{M}{m}\gamma)\subset \mathrm{Rep}(\varepsilon)$. For $t=\tau+\sum_0^{i-1}t_j$ $(\tau\in(0,t_i])$, since

$$d_2(\phi_2(\mu x_i,\sigma(x_i,\tau)),\phi_2(\mu x,\sigma(x,h(t)))\ )=$$

$$d_2(\phi_2(\mu x_i,\sigma(x_i,\tau)),\phi_2(\mu x,g(\textstyle\sum_0^{i-1}\sigma(x_j,t_j)+\sigma(x_i,\tau)))\ )\leq\gamma<\varepsilon',$$

we have that

$$d_1(x_0*t,x\cdot h(t))=d_1(\phi_1(x_i,\tau),\phi_1(x,h(t)))=$$

$$d_1(\mu^{-1}\phi_2(\mu x_i,\sigma(x_i,\tau)),\mu^{-1}\phi_2(\mu x,\sigma(x,h(t)))\ )\leq\varepsilon.$$

Therefore $\{x_i;t_i\}_0^k$ is ε-traced by (x,h) and the proof is completed.

Now we give the definition of suspension flow..

<u>Definition 5.3</u> ([4]). Let Y be a compact metric space with a metric ρ and $f:Y\longrightarrow Y$ a homeomorphism. Let $\alpha:Y\longrightarrow(0,\infty)$ be continuous. The suspension of f under α is the flow ϕ on the space

$$Y_\alpha=\bigcup_{0\leq t\leq\alpha(y)}(y,t)\Big/(y,\alpha(y))\sim(fy,0)$$

defined for small nonnegative time by $\phi((y,s),t)=(y,t+s)$, $0\le t+s<\alpha(y)$.

Each suspension space of f is homeomorphic to the suspension space of f under 1 (the constant function with value 1). A homeomorphism $\mu:Y_1 \longrightarrow Y_\alpha$ is given by $\mu(y,t)=(y,t\alpha(y))$. If a metric d of Y_1 is defined, a metric d_α of Y_α is induced by $d_\alpha((y_1,t_1),(y_2,t_2)) = d(\mu^{-1}(y_1,t_1),\mu^{-1}(y_2,t_2))$. For this reason we now define a metric on Y_1. Suppose the diameter of Y under ρ is less than 1.

Consider the subset $Y\times\{t\}$ of $Y\times[0,1]$ and let ρ_t denote the metric on $Y\times\{t\}$ defined by $\rho_t((y,t),(z,t))=(1-t)\rho(y,z)+t\rho(fy,fz)$ for $y,z\in Y$. Note that $\rho_0((y,0),(z,0))=\rho(y,z)$ and $\rho_1((y,1),(z,1))= \rho(fy,fz)$. Let $x_1,x_2\in Y_1$. Consider all finite chains $x_1=w_0,w_1,\ldots, w_n=x_2$ between x_1 and x_2 where for each i either w_i and w_{i+1} belong to $Y\times\{t\}$ for some t (in which case we call $[w_i,w_{i+1}]$ a horizontal segment) or w_i and w_{i+1} are on the same orbit (and then we call $[w_i,w_{i+1}]$ a vertical segment). Define the length of a chain to be the sum of the lengths of its segments where the length of a horizontal segment $[w_i,w_{i+1}]$ is measured in the metric ρ_t if w_i and w_{i+1} belong to $Y\times\{t\}$, and the length of a vertical segment $[w_i,w_{i+1}]$ is the shortest distance between w_1 and w_2 along the orbit (ignoring the direction on the orbit) using the usual metric on $\mathbb{R}$. If $w_i\neq w_{i+1}$ and w_i,w_{i+1} are on the same orbit and on the same set $Y\times\{t\}$ then the length of the segment $[w_i,w_{i+1}]$ is taken to be $\rho_t(w_i,w_{i+1})$, since this is always less than 1. Define $d(x_1,x_2)$ to be the infimum of the lengths of all chains between x_1 and x_2. Since $\rho_t((y,t),(z,t)) \ge \min\{\rho(y,z),\rho(fy,fz)\}$, it follows that $d(x_1,x_2)=0$ iff $x_1=x_2$. Clearly d is symmetric and satisfies the triangle inequality and hence is a metric on Y_1. Also d gives the topology on Y_1.

Lemma 5.4. Let f be a homeomorphism of a compact metric space Y and $\alpha: Y \longrightarrow (0,\infty)$ be a continuous map. Let (Y_α, ϕ_α) and (Y_1, ϕ_1) be suspensions of f under α and 1 respectively. Then (Y_α, ϕ_α) and (Y_1, ϕ_1) are topologically Lipschitz equivalent.

Proof. We define a homeomorphism $\mu: Y_1 \longrightarrow Y_\alpha$ by $\mu(y,\tau)=(y,\tau\alpha(y))$ for $(y,\tau)\in Y_1$ and define a continuous map $\sigma: Y_1 \times \mathbb{R} \longrightarrow \mathbb{R}$ by, for $(y,\tau)\in Y_1$ and $t\in\mathbb{R}$,

$$\sigma((y,\tau),t)=\begin{cases} \sum_{j=0}^{n-1}\alpha(f^j y)+\alpha(f^n y)(\tau+t-n)-\alpha(y)\tau & (\tau+t\in[n,n+1),n\geqq 0\ ) \\ -\sum_{j=n}^{-1}\alpha(f^j y)+\alpha(f^n y)(\tau+t-n)-\alpha(y)\tau & (\tau+t\in[n,n+1),n<0\ ). \end{cases}$$

Putting $m=\inf_{y\in Y}\alpha(y)$ and $M=\sup_{y\in Y}\alpha(y)$, we have

$0<m \leqq \dfrac{\sigma(x,t)-\sigma(x,s)}{t-s} \leqq M \quad (x\in Y_1,\ s\neq t)$ and $\sigma(x,0)=0 \quad (x\in Y_1)$. For every $(y,\tau)\in Y_1$ and every $t\in\mathbb{R}$, if $\tau+t\in[n,n+1)$ for some $n\in\mathbb{Z}$, then

$$\mu\phi_1((y,\tau),t)=\mu\phi_1((f^n y,0),\tau+t-n)=\phi_\alpha((f^n y,0),\alpha(f^n y)(\tau+t-n))$$

$$=\phi_\alpha(y,\sigma((y,\tau),t)+\alpha(y)\tau)=\phi_\alpha((y,\alpha(y)\tau),\sigma((y,\tau),t))$$

$$=\phi_\alpha(\mu(y,\tau),\sigma((y,\tau),t)).$$

Therefore (Y_α,ϕ_α) and (Y_1,ϕ_1) are topologically Lipschitz equivalent.

We recall that (Y,f) (a pair of compact metric space Y and homeomorphism f of Y) has the P.O.T.P. if for every $\varepsilon>0$ there is $\delta>0$ such that every δ-pseudo orbit $\{y_i\}_{i\in\mathbb{Z}}$ (i.e. $\rho(fy_i,y_{i+1})\leqq\delta$) has a point $y\in Y$ with $\rho(f^i y,y_i)\leqq\varepsilon \quad (i\in\mathbb{Z})$.

Theorem 5. Let f be a homeomorphism of a compact metric space Y and (Y_α,ϕ_α) a suspension of f under a continuous map $\alpha: Y\longrightarrow(0,\infty)$. (Y_α,ϕ_α) has the P.O.T.P. if and only if (Y,f) has the P.O.T.P..

Proof. By Theorem 4 and Lemma 5.4, we need only show the resulte when $\alpha=1$. Let ϕ denote the suspension of f under 1 and

$X = Y_1$, which has the suspension metric d induced by a metric ρ of Y. We suppose $Y = Y\times\{0\}\subset X$. For $W = Y\cdot[-1/4,1/4]$, we define a continuous map $\zeta:W\longrightarrow Y$ by $\zeta(y\cdot t) = y$ $(y\cdot t\in W)$. Put $U_0(\varepsilon) = \{x\in X: d(x,Y)\leq\varepsilon\}$ and $U_1(\varepsilon) = \{x\in X: d(x,Y\times\{1/2\})\leq\varepsilon\}$ for $\varepsilon>0$.

Suppose $(X,\emptyset)$ has the P.O.T.P.. Let $\varepsilon>0$ be given. There is $\varepsilon>\varepsilon_0>0$ such that $U_0(\varepsilon_0)\subset W$ and $U_0(\varepsilon_0)\cap U_1(\varepsilon_0)$ = empty, and such that if $d(x,x')\leq\varepsilon_0$ for $x,x'\in W$ then $\rho(\zeta(x),\zeta(x'))\leq\varepsilon$. By Theorem 1(b) there is $\delta>0$ so that every $(\delta,1)$-chain is ε_0-traced. Let $\{y_i\}_0^k$ be a finite δ-pseudo orbit of f. By Lemma 8 in [13], it is enough to show that there is $y\in Y$ with $d(f^i y,y_i)\leq\varepsilon$ $(0\leq i\leq k)$. For $t_i = 1$ $(0\leq i\leq k)$, $\{y_i;t_i\}_0^k$ is a $(\delta,1)$-chain, hence there are $x\in X$ and $g\in \mathrm{Rep}(\varepsilon)$ such that $d(y_i\cdot t,x\cdot g(i+t))\leq\varepsilon_0$ $(t\in[0,1],\ 0\leq i\leq k)$. Since $d(y_i,x\cdot g(i))\leq\varepsilon_0$, we have $x\cdot g(i)\in U_0(\varepsilon_0)$ $(0\leq i\leq k)$. Put $y = \zeta(x)$, then $d(y_0,y) = d(\zeta(y_0),\zeta(x)) \leq \varepsilon$. We claim that if $\zeta(x\cdot g(i)) = f^i y$ then $\zeta(x\cdot g(i+1)) = f^{i+1}y$ $(0\leq i\leq k-1)$. To see this, put $z = f^i y$. Assume $\zeta(x\cdot g(i+1)) \neq fz$, then there is $n\geq 2$ so that $\zeta(x\cdot g(i+1)) = f^n z$. Hence there is $s_1\in(0,1)$ such that $x\cdot g(i+s_1) = fz$. Since $d(y_i\cdot s_1,fz) = d(y_i\cdot s_1,x\cdot g(i+s_1)) \leq \varepsilon_0$, we get $y_i\cdot s_1\in U_0(\varepsilon_0)$, and so $s_1\in(0,1/4]$ or $s_1\in[3/4,1)$. When $s_1\in(0,1/4]$, since $d(y_i\cdot s_0,z\cdot(1/2)) \leq \varepsilon_0$ for some $0 < s_0 < s_1$, we have $y_i\cdot s_0\in U_1(\varepsilon_0)$. But, since $y_i\cdot s_0\in y_i\cdot[0,s_1]\subset U_0(\varepsilon_0)$, this contradicts the choice of ε_0. When $s_1\in[3/4,1)$, we get similarly the contradiction. Thus $\zeta(x\cdot g(i)) = f^i y$ $(0\leq i\leq k)$. Therefore we have $\rho(y_i,f^i y) = \rho(\zeta(y_i),\zeta(x\cdot g(i))) \leq \varepsilon$ $(0\leq i\leq k)$.

Conversely, suppose (Y,f) has the P.O.T.P.. Let $\varepsilon>0$ be given. There is $\varepsilon>\gamma>0$ such that $d(x,x')\leqq\gamma$ implies $d(x\cdot t,x'\cdot t)\leqq\varepsilon/3$ ($|t|\leqq 2$), and that $d(x\cdot t,x)\leqq\varepsilon/3$ ($|t|\leqq\gamma$). We take $\gamma>\delta>0$ and $\gamma/9>\varkappa>0$ such that δ-pseudo orbit of f is γ-traced, and if $h\in\mathrm{Rep}(6\varkappa)$ then $g\circ h\in\mathrm{Rep}(2\varepsilon)$ ($g\in\mathrm{Rep}(\varepsilon)$). There is $\delta_1>0$ with $\delta_1/(1-\delta_1)<\gamma$ such that $d((y,s),(y',s'))\leqq$ ($s,s'\in[0,1-\varkappa]$) implies $\rho(y,y')\leqq\delta$. Put $W_j=Y\cdot[-j\varkappa,j\varkappa]$ ($1\leqq j\leqq 3$) and take $\delta_0>0$ so that $B(W_j,\delta_0)\subset W_{j+1}$ ($j=1,2$) and $\zeta(B(x,\delta_0)\cap W)\subset B(\zeta(x),\delta_1)$ ($x\in W$). Put $T=1+6\varkappa$. Let a (δ_0,T)-chain $\Gamma_0=\{z_i;s_i\}_0^k$ be given. We put $x_0=z_0$, and define recursively $x_i\in Y\cup(X-W_2)$ ($1\leqq i\leqq k$) by $x_i=\zeta(z_i)$ if $z_i\in W_2$, or if $z_i\in W_3-W_2$ and $z_{i-1}\cdot s_{i-1}\in W_2$, and by $x_i=z_i$ otherwise. Then there is $t_i\geqq 1$ such that $x_i\cdot t_i\in Y\cup(X-W_2)$, $|s_i-t_i|\leqq 6\varkappa$, and that $\Gamma=\{x_i;t_i\}_0^k$ is a $(\delta_1,1)$-chain. Suppose Γ is ε-traced by (x,g). To get $h\in\mathrm{Rep}(6\varkappa)$, we define $h(0)=0$, $h(\sum_0^i s_j)=\sum_0^i t_j$ ($0\leqq i\leqq k$), and extend h linearly between these points. By choice of $\varkappa$ and γ, it follows that Γ_0 is 2ε-traced by $(x,g\circ h)$. Hence it remains to see that Γ_0 is ε-traced. There are $y_i\in Y$, $\tau_i,\tau_i'\in\{0\}\cup[\varkappa,1-\varkappa]$ and an integer $n_i\geqq 0$ such that $x_i=(y_i,1-\tau_i)$ and $t_i=\tau_i+n_i-1+\tau_i'$ ($0\leqq i\leqq k$). Then $y_i=x_i\cdot(\tau_i-1)$ and $f^{n_i}y_i=x_i\cdot(t_i-\tau_i')$. By choice of δ_1, we have $\rho(f^{n_i}y_i,y_{i+1})\leqq\delta$, so that $G=\{y_0,fy_0,\ldots,f^{n_0-1}y_0,y_1,\ldots,f^{n_k}y_k\}$ is a δ-pseudo orbit. Hence there is $y\in Y$ which γ-traces G. Put $N_i=\sum_0^{i-1}n_j$ ($1\leqq i\leqq k+1$), $N_0=0$, $S_i=\sum_0^i t_j$ ($0\leqq i\leqq k$) and $x=y\cdot(1-\tau_0)$. We define a reparametrization g Rep by

$$g(t)=\begin{cases} t & (t<t_0-\tau_0') \\ (\tau_i'+\tau_{i+1})^{-1}(t-S_i+\tau_i')+(\tau_0-1)+N_{i+1} & (S_i-\tau_i'\leqq t<S_i+\tau_{i+1}) \\ t-\sum_0^i(\tau_j'+\tau_{j+1}-1) & (S_i+\tau_{i+1}\leqq t<S_{i+1}-\tau_{i+1}') \\ t-\sum_0^{k-1}(\tau_j'+\tau_{j+1}-1) & (S_k-\tau_k\leqq t) \end{cases}$$

where $i=0,1,\ldots,k-1$. Since $|\tau_i'+\tau_{i+1}-1|\leqq d((f^{n_i}y_i,\tau_i'),(y_{i+1},1-\tau_{i+1}))$

$= d(x_i \cdot t_i, x_{i+1}) \leq \delta_1$, we get for $t, s \in \mathbb{R}$ $(t \neq s)$,

$$\left|\frac{g(t)-g(s)}{t-s} - 1\right| \leq \max_{0\leq i\leq k-1} \left|\frac{1}{\tau_i'+\tau_{i+1}} - 1\right| \leq \delta_1/(1-\delta_1) \leq \gamma.$$

Hence $g \in \mathrm{Rep}(\gamma) \subset \mathrm{Rep}(\varepsilon)$.

Let $t \in [0, S_k]$ be given. We show $d(x_0 * t, x \cdot g(t)) \leq \varepsilon$.

Case (1): $0 \leq t \leq t_0 - \tau_0'$. t is expressed as $t+1-\tau_0 = m+\tau$ for some integer $m \geq 0$ and $\tau \in [0,1)$. Then we have $d(x_0 * t, x \cdot g(t)) =$

$$d(x_0 \cdot (\tau_0 - 1 + m + \tau), x \cdot (\tau_0 - 1 + m + \tau)) = d((f^m y_0) \cdot \tau, (f^m y) \cdot \tau) < \varepsilon/3$$

because $\rho(f^m y_0, f^m y) \leq \gamma$.

Case (2): $S_i + \tau_{i+1} \leq t \leq S_{i+1} - \tau_{i+1}'$ $(0 \leq i \leq k-1)$. t is expressed as $t - S_i - \tau_{i+1} = m+\tau$ for some $m \geq 0$ and $\tau \in [0,1)$. Then we have

$$\begin{aligned}
&d(x_0 * t, x \cdot g(t)) \\
&= d(x_{i+1} * (\tau_{i+1} + m + \tau), x \cdot (m + \tau + S_i + \tau_{i+1} - \textstyle\sum_0^i (\tau_j' + \tau_{j+1} - 1))) \\
&= d(y_{i+1} \cdot (1 + m + \tau), y \cdot (1 + N_{i+1} + m + \tau)) \\
&= d((f^{m+1} y_{i+1}) \cdot \tau, (f^{m+1+N_{i+1}} y) \cdot \tau) < \varepsilon/3
\end{aligned}$$

because $\rho(f^m y_{i+1}, f^{m+N_{i+1}} y) \leq \gamma$.

Case (3): $S_i - \tau_i' \leq t \leq S_i + \tau_{i+1}$ $(0 \leq i \leq k-1)$. Then $t = \tau + S_i - \tau_i'$ for some τ with $0 \leq \tau \leq \tau_i' + \tau_{i+1} \leq 1 + \delta_1 \leq 2$. We get

$$\begin{aligned}
&d(x_0 * t, x \cdot g(t)) \\
&\leq d(x_{i+1} * (\tau - \tau_i'), x \cdot (g(S_i - \tau_i') + \tau)) + d(x \cdot (g(S_i - \tau_i') + \tau), x \cdot g(S_i - \tau_i' + \tau))
\end{aligned}$$

$\leq d((f^{n_i}y_i)\cdot\tau,(f^{N_{i+1}}y)\cdot\tau)+\varepsilon/3$ (since $|g(S_i-\tau_i')+\tau-g(S_i-\tau_i'+\tau)|\leq\gamma$)

$\leq d((f^{n_i}y_i)\cdot\tau,y_{i+1}\cdot\tau)+d(y_{i+1}\cdot\tau,(f^{N_{i+1}}y)\cdot\tau)+\varepsilon/3$

$\leq\varepsilon/3+\varepsilon/3+\varepsilon/3=\varepsilon$

because $\rho(f^{n_i}y_i,y_{i+1})\leq\gamma$, $\rho(y_{i+1},f^{N_{i+1}}y)\leq\gamma$ and $|\tau|\leq 2$.

Case (4): $S_k-\tau_k'\leq t$. t is expressed as $t=S_k+\tau-\tau_k'$ for some $0\leq\tau\leq\tau_k'$. Then we have

$$d(x_0*t,x\cdot g(t))\leq d(x_k\cdot(t_k+\tau-\tau_k'),x\cdot(\tau-\tau_k'+S_k-\textstyle\sum_0^{k-1}(\tau_j'+\tau_{j+1}-1)))$$
$$=d((f^{n_k+1}y_k)\cdot\tau,(f^{n_k+1+N_k}y)\cdot\tau)\leq\varepsilon/3$$

because $\rho(f^{n_k+1}y_k,f^{n_k+1+N_k}y)\leq\gamma$.

Therefore the finite $(\delta_1,1)$-chain $\{x_i;t_i\}_0^k$ is ε-traced and by Theorem 1(c) we have that $(X,\emptyset)$ has the P.O.T.P.. The proof is completed.

§ 6 Isometric flows.

Let X be a compact metric space. A flow $(X,\emptyset)$ is isometry if there is a metric d on X with $d(x\cdot t,y\cdot t)=d(x,y)$ $(x,y\in X,\ t\in\mathbb{R})$. $(X,\emptyset)$ is minimal if for every $x\in X$ the orbit of x is dense in X.

Theorem 6. Let M be a compact connected manifold and $\emptyset$ an isometric flow on M with respect to a Riemannian metric d. If $(M,\emptyset)$ has the P.O.T.P., then $(M,\emptyset)$ is minimal.

Proof. Assume that $(M,\emptyset)$ is not minimal. We will show that $(M,\emptyset)$ does not have the P.O.T.P.. Since $\emptyset$ is isometry, $\emptyset$ preserves the measure μ on M induced by a Riemannian metric d and so the set $Q=\{y\in M$: for every $\varepsilon>0$ there is a sequence $t_n\uparrow\infty$ with $d(y,y\cdot t_n)<\varepsilon\}$

is dense; this follows from Birkoff's return theorem [9].
By assumption there is $x\in M$ such that the orbit of x, $O(x)$, is not dense in M. Since Q is dense in M, there are $z\in Q$ and $\varepsilon>0$ so that $V\cap B=$ empty for $V=\{y\in M: d(\overline{O(x)},y)\leq\varepsilon\}$ and $B=\{y\in M: d(z,y)\leq\varepsilon\}$. Assume that (M,ϕ) has the P.O.T.P.. Then there are $\delta>0$ and $T>0$ such that (δ,T)-chain is ε-traced. Since M is connected, we can find a sequence $\{x_i\}_0^k$ such that $x_0=z$, $x_k=x$ and $x_i\in Q$ $(0\leq i\leq k-1)$, and such that $d(x_i,x_{i+1})<\delta/2$ $(0\leq i\leq k-1)$ and $\{x_i:0\leq i\leq k\}$ is $\delta/2$-dense (i.e. $\{y\in M: d(y,x_i)<\delta/2$ for some x_i, $0\leq i\leq k\}=M$). As $x_i\in Q$, there is $t_i\geq T$ with $d(x_i\cdot t_i,x_i)<\delta/2$ $(0\leq i\leq k-1)$. Put $t_k=0$. Then $\{x_i;t_i\}_0^k$ is a (δ,T)-chain, hence there are $y\in M$ and $g\in Rep(\varepsilon)$ with $d(x_0*t,y\cdot g(t))\leq\varepsilon$ for $t\in[0,\sum_0^{k-1}t_i]$. For $K=g(\sum_0^{k-1}t_i)$, since $d(x,y\cdot K)=d(x_k,y\cdot K)\leq\varepsilon$, we have $d(\overline{O(x)},y)\leq d(x\cdot(-K),y)=d(x,y\cdot K)\leq\varepsilon$; i.e. $y\in V$. On the other hand, since $d(z,y)\leq\varepsilon$, we get $y\in B$, hence $V\cap B$ is not empty. This is a contradiction as well.

Remark 1. Let f be a homeomorphism of a compact connected manifold M. Using minimality of f, A.Morimoto showed that if f is isometry and the dimension of $M\geqq 1$ then f has not P.O.T.P. [10]. This result was generalized in the case of distal homeomorphisms on compact connected metric spaces by N.Aoki[1].

Remark 2. In Theorem 6, the converse is not true. There is an isometric minimal flow which does not have the P.O.T.P.. For example, let f be an irrational rotation (orientation preserving) homeomorphism of the circle S^1. Then the suspension space of (S^1,f) under 1 is just 2-torus T^2 and the induced flow ϕ is isometric and minimal. But (T^2,ϕ) does not have the P.O.T.P., because (S^1,f) does not have the P.O.T.P. (see Remark 5) and Theorem 5 holds.

<u>Remark 3.</u> There is an isometric minimal flow on S^1 which has the P.O.T.P.. Indeed, let ϕ be a constant rotation flow on S^1. Since (S^1,ϕ) is a suspension of the identity homeomorphism of a single point space; $(\{p\},id.)$, (S^1,ϕ) has the P.O.T.P. by Theorem 5 (it is clear that $(\{p\},id.)$ has the P.O.T.P.).

<u>Remark 4.</u> In general, the direct product of flows with the P.O.T.P. need not to have the P.O.T.P.. For example, let (S^1,ϕ) be a constant rotation flow. This has the P.O.T.P. as in Remark 3, but the direct product flow $(S^1\times S^1,\phi\times\phi)$ (i.e. $(\phi\times\phi)((x_1,x_2),t) = (\phi(x_1,t),\phi(x_2,t))$, $(x_1,x_2)\in S^1\times S^1$, $t\in\mathbb{R}$) dose not have the P.O.T.P. by Theorem 6, because $\phi\times\phi$ is isometry and is not minimal.

<u>Remark 5.</u> Let $\phi:X\times\mathbb{R}\longrightarrow X$ be a flow on a compact metric space X. A time T-map of ϕ is a homeomorphism of X defined by $f_T=\phi(\cdot,T):X\longrightarrow X$. In general, a time T-map of flow with the P.O.T.P. need not to have the P.O.T.P.. For example, let $S^1 = \{z\in\mathbb{C}: |z|=1\}$, $d(z_1,z_2)=|z_1-z_2|$ $(z_1,z_2\in S^1)$ and $\phi(z,t)=ze^{2\pi it}$ $(z\in S^1,\ t\in\mathbb{R})$ where $i=\sqrt{-1}$. Then (S^1,ϕ) has the P.O.T.P. as in Remark 3. The time T-map f_T of ϕ is given by $f_T(z) = ze^{2\pi iT}$. f_T dose not have the P.O.T.P.. Indeed, for every $1/2>\delta>0$, $\{e^{2\pi in(T+(\delta/2))}\}_{n\geq 0}$ is a δ-pseudo orbit of f_T. But there is no point which can 1/2-trace this δ-pseudo orbit.

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